



Lecture 11 (spin) quantum spin and Hahn echo

①

NMR (nuclear magnetic resonance)

$H = -\vec{\mu} \cdot \vec{B}$
 $\vec{\mu} = \gamma_p \vec{S}$
 $(\vec{\mu}_e = -\gamma_e \vec{S})$

$\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B}$

②

quantum spin
||
two-level system
(qubit)

$|\psi\rangle = c_g |g\rangle + c_e |e\rangle \rightarrow \theta, \phi$
 $|c_g|^2 + |c_e|^2 = 1$

$c_g = \cos(\theta/2) e^{i\phi_g}$
 $c_e = \sin(\theta/2) e^{i\phi_e}$

$\phi = \phi_e - \phi_g$
 $e^{-i(\phi_e - \phi_g)}$
 $e^{i(\phi_e - \phi_g)}$
 phase sensitive

$c_g(t) = A c_g(0) + B c_e(0)$
 $c_e(t) = A' c_g(0) + B' c_e(0)$

$P_e(t) = |c_e(t)|^2$
 $= |A c_g(0) + B c_e(0)|^2$
 $= |A c_g(0)|^2 + |B c_e(0)|^2 + (A^* B^* c_g^*(0) c_e(0) + A^* B c_g(0) c_e^*(0))$
 $= |A c_g(0)|^2 + |B c_e(0)|^2 + (A^* B^* c_g^*(0) c_e(0) + A^* B c_g(0) c_e^*(0))$





✓ quantum spin
- optical Bloch equation
- Hahn spin echo

Rabi model

$0 < t < \tau$

$E = E_0 \cos(\omega t - \phi)$

Lab frame $|\psi\rangle = a_g |g\rangle + a_e |e\rangle$
 $\downarrow$
 $a_g = C_g$
 $a_e = C_e e^{-i\omega t}$

rotating frame (ω) $|\psi\rangle_{\text{rot}} = C_g |g\rangle + C_e |e\rangle = \begin{pmatrix} C_g \\ C_e \end{pmatrix}$

$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$

$\frac{d}{dt} \begin{pmatrix} a_g \\ a_e \end{pmatrix} = \left[\begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \begin{pmatrix} 0 & V_{ge} \\ V_{eg} & 0 \end{pmatrix} \right] \begin{pmatrix} a_g \\ a_e \end{pmatrix}$

$\omega_g = E_g / \hbar$
 $\omega_e = E_e / \hbar$

rotating frame (ω)
 $\downarrow$
 $\hat{H}_{\text{rot}}$

$\begin{cases} i \frac{d}{dt} C_g = -\frac{\chi}{2} e^{i(\omega - \omega_g)t} e^{-i\phi} C_e \\ i \frac{d}{dt} C_e = -\frac{\chi}{2} e^{-i(\omega - \omega_g)t} e^{i\phi} C_g \end{cases}$

$\chi \equiv e \hbar \vec{x}_g \cdot \vec{E}_0$

④

$\hat{\rho} = |\psi\rangle\langle\psi|$

$= \begin{pmatrix} C_g \\ C_e \end{pmatrix} \begin{pmatrix} C_g^* & C_e^* \end{pmatrix}$

$= \begin{pmatrix} C_g C_g^* & C_g C_e^* \\ C_e C_g^* & C_e C_e^* \end{pmatrix}$

$= \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}$

$\rho_{gg} + \rho_{ee} = 1$

$\frac{d\rho_{gg}}{dt} = \dot{C}_g C_g^* + C_g \dot{C}_g^*$

$= C_e \left(\frac{i\chi}{2} e^{i\Delta t} e^{-i\phi} C_e \right)^* + \left(\frac{i\chi}{2} e^{-i\Delta t} e^{i\phi} C_g \right) C_g^*$

$= \frac{i\chi}{2} e^{-i\Delta t} e^{i\phi} (|C_g|^2 - |C_e|^2)$

$= \frac{i\chi}{2} e^{-i\Delta t} e^{i\phi} (\rho_{gg} - \rho_{ee})$

$\frac{d\rho_{ge}}{dt} = \dot{C}_g C_e^* + C_g \dot{C}_e^*$

$= \frac{i\chi}{2} e^{-i\Delta t} e^{i\phi} \rho_{ge} + \frac{i\chi}{2} e^{i\Delta t} e^{-i\phi} \rho_{eg}$

rotating frame (ω)
 $\downarrow$
 $\omega - \omega_g$

$\rho_{ge} = \rho_{ge} e^{-i\Delta t}$
 $\rho_{eg} = \rho_{eg} e^{i\Delta t}$

Self-term is observable

cross-term in observables

$\frac{d\rho_{ge}}{dt} = -\frac{d\rho_{eg}}{dt}$





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NMR
 $\frac{d\vec{C}}{dt} = \vec{r} \times \vec{B}$

density matrix in rotating frame (ω)

$$\hat{\rho} = \begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}$$

$$= \frac{i}{2} (u \hat{\sigma}_x + v \hat{\sigma}_y + w \hat{\sigma}_z + \hat{I})$$

$$= \begin{pmatrix} \frac{1+w}{2} & \frac{u-iv}{2} \\ \frac{u+iv}{2} & \frac{1-w}{2} \end{pmatrix}$$

$$u = \rho_{ge} + \rho_{eg}$$

$$v = i(\rho_{ge} - \rho_{eg})$$

$$w = \rho_{gg} - \rho_{ee}$$

$$\frac{d\rho_{gg}}{dt} = i\Delta\rho_{eg} + \frac{i\chi}{2} e^{i\phi} (\rho_{ee} - \rho_{gg})$$

$$\frac{d\rho_{ge}}{dt} = -i\Delta\rho_{ge} - \frac{i\chi}{2} e^{-i\phi} (\rho_{gg} - \rho_{ee})$$

$$\frac{d\rho_{eg}}{dt} = \frac{i}{2} (\chi e^{i\phi} \rho_{ge} - \chi^* e^{-i\phi} \rho_{gg})$$

$$\frac{d\rho_{ee}}{dt} = \frac{i}{2} (\chi e^{i\phi} \rho_{ge} - \chi^* e^{-i\phi} \rho_{gg})$$

$$\vec{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{cases} \frac{du}{dt} = -\Delta v - \chi \sin(\phi + \omega t) u \\ \frac{dv}{dt} = \Delta u + \chi \cos(\phi + \omega t) u \end{cases}$$

$$\vec{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \rightarrow \begin{cases} \frac{du}{dt} = \chi \sin(\phi + \omega t) u - \chi \cos(\phi + \omega t) v \\ \frac{dv}{dt} = \chi \cos(\phi + \omega t) u + \chi \sin(\phi + \omega t) v \end{cases}$$

$$\vec{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \frac{dw}{dt} = \chi \sin(\phi + \omega t) u - \chi \cos(\phi + \omega t) v$$

$$\vec{B}_{eff} = (\chi \cos(\phi + \omega t), \chi \sin(\phi + \omega t), -\Delta)$$

⑥

$$u = \rho_{ge} e^{-i\omega t} + \rho_{eg} e^{i\omega t} = c_g c_e^* e^{-i\omega t} + c_e c_g^* e^{i\omega t} = \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) e^{-i\phi} e^{-i\omega t} + \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) e^{i\phi} e^{i\omega t}$$

$$v = i(\rho_{ge} e^{-i\omega t} - \rho_{eg} e^{i\omega t}) = i(c_g c_e^* e^{-i\omega t} - c_e c_g^* e^{i\omega t}) = i\left(\cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) e^{-i\phi} e^{-i\omega t} - \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) e^{i\phi} e^{i\omega t}\right)$$

$$w = \rho_{gg} - \rho_{ee} = |c_g|^2 - |c_e|^2 = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta$$

$$u = \frac{1}{2} \sin\theta (e^{-i(\phi+\omega t)} + e^{i(\phi+\omega t)}) = \sin\theta \cos(\phi + \omega t)$$

$$v = \frac{i}{2} \sin\theta (e^{-i(\phi+\omega t)} - e^{i(\phi+\omega t)}) = \sin\theta \sin(\phi + \omega t)$$

$$w = \cos\theta$$





